Perpetuants: a lost treasure

Claudio Procesi (Joint work with H. Kraft)

Claudio Procesi Roma, La Sapienza

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Back to 1882

Invariant Theory

Binary forms

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The theorem of Stroh

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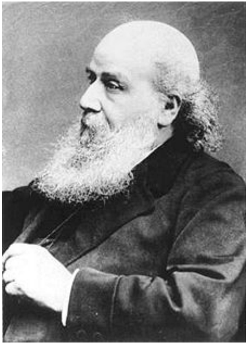
A sketch of the proof

Perpetuant

is one of the several concepts invented (in 1882) by J. J. Sylvester

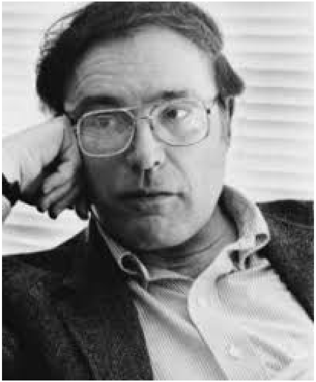


in his investigations of *covariants* for binary forms.

It appears in one of the first issues of the American Journal of Mathematics which he had founded a few years before.

Perpetuant is a concept of *Invariant theory*, and a name which will hardly appear in a mathematical paper of the last 70 years

I learned of this word from Gian-Carlo Rota who pronounced it with an enigmatic smile.



Invariant Theory

Algebraic invariant theory in its simplest form

treats of a group G of linear transformations on a vector space V.

- The action extends to polynomial functions $S[V^*]$ on V, by the formula

$$\boxed{(g \cdot f)(v) := f(g^{-1}v)}$$

- A polynomial $f(v)$ is *G invariant* if

$$(g \cdot f)(v) := f(g^{-1}v) = f(v), \quad \forall g \in G, \ v \in V.$$

- the invariants form a subalgebra of $S[V^*]$ denoted by $S[V^*]^G$.

Back to 1850

In 19^{th} century the theory was developed essentially for V the space of homogeneous polynomials of some degree q in k–variables and G the special linear group $SL(k, \mathbb{C})$ acting on these variables or for the direct sum of copies of such forms.
For $k = 2$ one then speaks of *binary quantics* or q–antics:

$$\boxed{f(x, y) = \sum_{i=0}^{q} a_i x^{q-i} y^i}$$ a general binary quantic.

$q = 2, 3, 4, 5, \cdots$ binary *quadratic, cubic, quartic, quintic, etc.*
The group acting is $SL(2, \mathbb{C})$ (acts on x, y).

- This is a $q + 1$ dimensional vector space V_q
- The polynomial functions over V_q are $\mathbb{C}[a_0, a_1, a_2, \ldots, a_q]$.

A quick course on binary forms

The invariants of a general binary quantic $\mathbb{C}[a_0, a_1, a_2, \ldots, a_q]^{SL(2,\mathbb{C})}$ are thus special polynomials in the variables $a_0, a_1, \ldots, a_q$.

The space of polynomials $\mathbb{C}[a_0, a_1, a_2, \ldots, a_q]$ can be bigraded by defining the *weight g* of a_i to be i so that

$$g(\prod_{j=0}^{q} a_j^{h_j}) = \sum_{j=1}^{q} j \cdot h_j.$$

A polynomial with terms all of the same weight is called *isobaric*.

a basic problem

of 19^{th} century invariant theory was to

1. describe a minimal set of generators for invariants
2. and possibly also a minimal set of relations.

Classically generators for $SL(2,\mathbb{C})$–invariants were computed for $q \leq 8$ (some gaps for $q = 7$).
Now with the help of computers a few other cases have been analyzed for $q \leq 12$.

$SL(2,\mathbb{C})$–invariants can be computed from U-invariants

Use the formalism of *divided powers*

$$\lambda^{[h]} := \frac{\lambda^h}{h!} \implies (\lambda+\mu)^{[h]} = \sum_{i=0}^{h} \lambda^{[h-i]}\mu^{[i]}.$$

Consider the polynomial rings $\cdots A(q) \subset A(q+1) \subset \cdots$

$$A(q) := \mathbb{C}[a_0, a_1, a_2, \ldots, a_q], \quad q = 0, \ldots, \infty$$

and the action on $A(q)$ of the additive group $\lambda \in \mathbb{C} = U$ subgroup of $SL(2,\mathbb{C})$

$$U := \left\{ \begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix}, \quad \lambda \in \mathbb{C} \right\}$$

induced by the action on the variables:

$$\begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x + \lambda y \\ y \end{vmatrix}.$$

A quick course on binary forms

We can see that the U action is (normalizing the coefficients):

$$\lambda \cdot a_0 = a_0, \quad \lambda \cdot a_1 = a_0\lambda + a_1, \quad \lambda \cdot a_2 = a_0\lambda^{[2]} + a_1\lambda + a_2,$$
$$\lambda \cdot a_3 = a_0\lambda^{[3]} + a_1\lambda^{[2]} + a_2\lambda + a_3,$$
$$\lambda \cdot a_4 = a_0\lambda^{[4]} + a_1\lambda^{[3]} + a_2\lambda^{[2]} + a_3\lambda + a_4, \ \ldots$$
$$\ldots\ldots\ldots$$
$$\lambda \cdot a_q = a_0\lambda^{[q]} + a_1\lambda^{[q-1]} + a_2\lambda^{[q-2]} + \cdots + a_{q-1}\lambda + a_q$$

U–invariants

denote the ring of U–invariants of binary forms of degree q (or polynomials) by

$$\boxed{S(q) = \mathbb{C}[a_0, a_1, a_2, \ldots, a_q]^U.}$$

- Then $S(q) = \oplus S(q)_{k,g}$ is bigraded, that is it decomposes into a direct sum of components, homogeneous of degree k and isobaric of weight g.
- The ring of invariants under $SL(2, \mathbb{C})$ is the subring of $S(q)$ direct sum of the homogeneous and isobaric components with $g = \frac{q \cdot k}{2}$.

A quick course on binary forms

That is a homogeneous invariant of a binary q–form is a U invariant with the relation between

the degree k and the weight g

$$g = \frac{q \cdot k}{2}.$$

As example the *discriminant* of the cubic ($q = 3$):

$$D = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2$$

of degree 4 and weight $6 = \frac{3 \cdot 4}{2}$, generates the algebra of invariants of the cubic. For $q = 2$ we have also the discriminant $a_1^2 - 2a_0a_2$.

Example of the cubic (new notations for a_i)

The algebra of U–invariants, for the cubic, is generated by 4 elements. The discriminant D, and a_0, H, T:

$$D = 9a_0^2a_3^2 - 18a_0a_1a_2a_3 + 8a_0a_2^3 + 6a_1^3a_3 - 3a_1^2a_2^2$$

$$a_0, \quad H = a_1^2 - 2a_0a_2, \quad T = a_1^3 - 3a_0a_1a_2 + 3a_0^2a_3$$

the element a_0 (degree 1, weight 0), and H of degree 2 and weight 2, T of degree 3 and weight 3.

They are related by the syzygy $H^3 + Da_0^2 - T^2 = 0$ of degree 6 and weight 6.

Example of the quartic

The algebra of U–invariants, for the quartic is generated by 5 elements a_0, B, C, H, T.

$$a_0, \quad H = a_1^2 - 2a_0a_2, \quad T = a_1^3 - 3a_0a_1a_2 + 3a_0^2a_3$$

$$B = 2a_0a_4 - 2a_1a_3 + a_2^2,$$

$$C = 2a_2^3 - 6a_1a_2a_3 + 9a_0a_3^2 + 6a_1^2a_4 - 12a_0a_2a_4.$$

Relation $\boxed{3a_0^2HB - a_0^3C - H^3 + T^2 = 0}$.

Notice that now $D = -3HB + a_0C$.

commutative algebra

in modern terms given a graded commutative algebra $A = \oplus_{i=0}^{\infty} A_i$ over a field $F = A_0$, setting $I = \oplus_{i=1}^{\infty} A_i$ we have that the elements of I^2 are *decomposable* and a minimal set of generators of A is a set of homogeneous elements giving a basis of I/I^2.

I/I^2 is a graded vector space and it is finite dimensional if and only if A is finitely generated over F.

Apply this to $A = S(q)$

and denote by I_q the subspace of U–invariants with no constant term.

Then I_q/I_q^2 is *bigraded* by degree and weight and a basic problem is to prove that it is finite dimensional and compute the dimension of its bigraded pieces.

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a basic problem

A lot of work by

Cayley, Sylvester, Clebsch, Gordan and others was devoted to compute U–invariants with success for forms of degree $q \leq 6$, there are 23 generators for $q = 5$ the largest of degree 18.

Recent work, with the help of computers, gives further results up to degree 12.

A crowning point of this research was Gordan's proof that these algebras of invariants are finitely generated, but no explicit formulas for either the generators or even for just the weight and degree of these generators is known for $q > 12$.

Hilbert

The theory was revolutionised by Hilbert at the end of the century.

He proved the finiteness theorem of forms in any number of variables and asked, in his 14^{th} problem, if the finiteness theorem is true for every group.

Negative answer by Nagata 1958.

We can finally define:

Perpetuants

a stable problem

From the formulas it is clear that the ring of U–invariants $S(q)$ is contained in the ring of U–invariants $S(q+1)$ and so on.

So one can define the ring of U–invariants $S = \bigcup_q S(q)$ as the subring of the polynomial ring:

$$\mathbb{C}[a_0, a_1, a_2, \ldots, a_n, \ldots], \quad a_i, \ i = 0, \ldots, \infty$$

in the infinitely many variables $a_i, \ i = 0, \ldots, \infty$ invariant under the limit action of the 1–parameter subgroup U.

a basic theorem

This 1–parameter subgroup $\lambda \cdot a_k = \sum_{j=0}^{k} a_j \lambda^{[k-j]}$,

has as infinitesimal generator the differential operator

$$\boldsymbol{D} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}, \quad \boldsymbol{D}(a_i) = a_{i-1}, \ \boldsymbol{D}(a_0) = 0.$$

For each $q = 0, 1, \ldots, \infty$ the algebra $S = \mathbb{C}[a_0, a_1, a_2, \ldots . a_q]^U$ of U-invariants is formed by the polynomials f in the variables $a_0, a_1, a_2, \ldots, a_q$, satisfying $\boldsymbol{D} f = 0$ i.e.:

$$\sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} f(a_0, a_1, a_2, \ldots) = 0, \quad \boldsymbol{D} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}.$$

Perpetuants

It was quickly discovered that

- an element of $S(q)$, which is indecomposable in $S(q)$, need not remain indecomposable in $S(q+1)$.
- In other words: the maps $I_q/I_q^2 \to I_{q+1}/I_{q+1}^2$ need not be injective,
- or also: a minimal set of generators for $S(q)$ cannot be completed to one for $S(q+1)$.

As an example the generator D for $S(3)$ is decomposable in $S(4)$.

$$D = -3HB + a_0 C.$$

Perpetuants

Definition

A *perpetuant* is an indecomposable element of $S(q)$ which remains indecomposable in all $S(k),\ k \geq q$ hence in $S = \bigcup S(k)$.

In other words it gives an element of I_q/I_q^2 which *lives forever*, that is it remains nonzero in all $I_k/I_k^2,\ \forall\ k \geq q$.

In this sense it is *perpetuant*.

In other words, denoting by $I \subset S$ the ideal of positive elements of S *perpetuants* are essentially the elements of $I \setminus I^2$.

Perpetuants

Thus to describe perpetuants is strictly related to describe minimal sets of generators for the graded algebra $S = \mathbb{C}[a_0, a_1, a_2, \ldots]^U$.

In other words, denoting by $I \subset S$ the ideal of positive elements of S we want to describe I/I^2.

This space decomposes into a direct sum

$$I/I^2 = \bigoplus_{n,g \in \mathbb{N}} P_{n,g}$$

with $P_{n,g}$ the image of the elements in I of degree n and weight g.

Perpetuants

In our main theorem, see page ??

we exhibit a space of perpetuants that is a bigraded subspace

$$\Pi = \oplus_{i,g} \Pi_{i,g} \subset I$$

which maps isomorphically to

$$I/I^2 = \bigoplus_{n,g \in \mathbb{N}} P_{n,g}.$$

.

A basis of a space of perpetuants is thus a minimal set of generators for the algebra S of U invariants.

The theorem of Emile Stroh

conjectured by Mac Mahon and proved by Stroh

$$\sum_{g=0}^{\infty} \dim(P_{n,g})\, x^g = \begin{cases} \dfrac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)} & \text{for } n > 2, \\[2ex] \dfrac{x^2}{(1-x^2)} & \text{for } n = 2, \\[2ex] 1 & \text{for } n = 1. \end{cases}$$

For $n \leq 3$ proved by Sylvester.

Our Main Theorem, *development of Stroh*

we use the partial order

$$(t_2, \ldots, t_n) \succeq (s_2, \ldots, s_n) \iff t_i \geq s_i \text{ for all } i.$$

Theorem

The elements $U_{\mathbf{k}} = U_{k_2,\ldots,k_n} = \tilde{U}_{0,k_2,\ldots,k_n}$ *with* $\sum_{i=2}^{n} i \cdot k_i = g$ *and*

$$\mathbf{k} \succeq \mathbf{n} = (0, 2^{n-4}, 2^{n-5}, \ldots, 4, 2, 1, 1), \quad (resp.\ \mathbf{n} = (0, 1))$$

form a basis of a space of perpetuants of degree $n > 3$ *(resp.* $n = 3$*) and weight* g.

For $n = 2$ *one perpetuant,* U_{2k}*, in each even weight* $2k > 0$.
For $n = 1$ *just* a_0.

Notice

it is a remarkable fact that

we are unable to exhibit minimal sets of generators for U invariants of forms of a given degree q but in fact we are able to do this for the limit, infinite, case.

The use of symmetric functions

Take variables $\lambda_1, \ldots, \lambda_n$ and let S_n be the symmetric group permuting the λ_i.

Definition

Denote by $\Sigma_{n,g} \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ the subspace of *symmetric polynomials* in $\lambda_1, \ldots, \lambda_n$ which are homogeneous of degree g.

This space has different combinatorial bases all indexed by partitions of g into n parts.

Many are in fact bases for $\mathbb{Z}[\lambda_1, \ldots, \lambda_n]^{S_n}$.

The use of symmetric functions

We first take as basis of $\Sigma_{n,g}$ the simplest:

total monomial sums $m_{h_1,\dots,h_n}$

i.e. the sum over the $\mathcal{S}_n$-orbit of $\lambda_1^{h_1}\cdots\lambda_n^{h_n}$ where $h_1 \geq h_2 \geq \cdots \geq h_n \geq 0$ and $h_1 + \cdots + h_n = g$:

$$m_{h_1,\dots,h_n}(\lambda) := \sum_{\mathcal{S}_n\text{-orbit}} \lambda_{\sigma(1)}^{h_1}\cdots\lambda_{\sigma(n)}^{h_n}.$$

The use of symmetric functions

Another basis of the space $\Sigma_{n,g}$ of symmetric functions is formed by the monomials

$$\mathrm{e}_1^{k_1}\dots\mathrm{e}_n^{k_n},\quad \sum_j jk_j=g,\quad \prod_{i=1}^n(t+\lambda_i)=t^n+\sum_{i=1}^n t^{n-i}\mathrm{e}_i$$

where e_i is the i th elementary symmetric function.

The base change:

$$m_{h_1,\dots,h_n}=\sum_{k_1,\dots,k_n}\alpha_{h_1,\dots,h_n;k_1,\dots,k_n}\mathrm{e}_1^{k_1}\dots\mathrm{e}_n^{k_n}. \tag{1}$$

The $\alpha_{h_1,\dots,h_n,k_1,\dots,k_n}$ are computable integers obtained inverting the obvious expansion

$$\mathrm{e}_1^{k_1}\dots\mathrm{e}_n^{k_n}=\sum_{h_1,\dots,h_n}\beta_{h_1,\dots,h_n;k_1,\dots,k_n}m_{h_1,\dots,h_n}. \tag{2}$$

The use of symmetric functions

Let $A_{n,g}$ denote the subspace of $\mathbb{C}[a_0, a_1, a_2, \ldots, a_n, \ldots]$ of elements of degree n and weight g

Define the polynomials $\tilde{U}_{k_1,\ldots,k_n} \in A_{n,g}$ by the formula:

$$\tilde{U}_{k_1,\ldots,k_n} = \sum_{h_1,\ldots,h_n} \alpha_{h_1,\ldots,h_n;k_1,\ldots,k_n} \prod_{j=1}^{n} a_{h_j},$$

Observe that the polynomials $\tilde{U}_{k_1,\ldots,k_n}$ form a basis of $A_{n,g}$

A basis of the U-invariants

The first main theorem is the formula:

$$\boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \begin{cases} 0 & \text{if } k_1 = 0 \\ \tilde{U}_{k_1-1,\dots,k_n} & \text{if } k_1 > 0. \end{cases}$$

Example

$$\boldsymbol{D}\tilde{U}_{0,2} = \boldsymbol{D}(2a_0a_4 - 2a_1a_3 + a_2^2) = 0.$$

This implies the Theorem

The elements $U_{k_2,\dots,k_n} := \tilde{U}_{0,k_2,\dots,k_n}$, $\sum_{i=2}^{n} ik_i = g$ form a basis of the space $S_{n,g}$ of the U-invariants of degree n and weight g.

A duality for the U-invariants

Corollary

$$\sum_{g=0}^{\infty} \dim(\overline{\Sigma}_{n,g})\, x^g = \sum_{g=0}^{\infty} \dim(S_{n,g})\, x^g = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}$$

The partition function

Notice that in the series

$$\sum_{i=0}^{\infty} p_i(n)x^i = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}$$

the integer $p_i(n)$ counts the *number of ways in which the integer i can be written as a sum of integers* $2, 3, \ldots, n$ hard to compute!.

In the movie "The Man Who Knew Infinity" there is a competition between Mac Mahon and Srinivasa Ramanujan, to compute $p_{100}(100)$*!*

Mac Mahon and Srinivasa Ramanujan

The potenziante and umbral calculus

The proof of the formula of Mac Mahon umbral calculus

We define a linear map $\boldsymbol{E}$ from the space of all polynomials in auxiliary variables $\alpha_1, \ldots, \alpha_n$ (*the umbrae*) to the space of polynomials of degree n in the variables $a_0, a_1, a_2, \ldots$,

$$\boldsymbol{E} \colon \mathbb{C}[\alpha_1, \ldots, \alpha_n] \to \mathbb{C}[a_0, a_1, a_2, \ldots], \ \alpha_1^{[r_1]} \cdots \alpha_n^{[r_n]} \overset{\boldsymbol{E}}{\mapsto} a_{r_1} \cdots a_{r_n},$$

1. a homogeneous polynomial of degree g, in the $\alpha_i, \ i = 1, \ldots, n$, is mapped to an isobaric polynomial of weight g in the a_j, and homogeneous of degree n.
2. The map $\boldsymbol{E}$ commutes with the permutation action on the α_i, $i = 1, \ldots, n$

$$\boldsymbol{E}(\alpha_1^{[3]}\alpha_2^{[2]}) = \boldsymbol{E}(\alpha_3^{[3]}\alpha_1^{[2]}) = \boldsymbol{E}(\alpha_3^{[3]}\alpha_1^{[2]} \prod_{j\neq 1,3} a_j^{[0]}) = a_0^{n-2}a_2a_3$$

$$\boldsymbol{E}(\alpha_i^{[2]}\alpha_j^{[2]}) = \boldsymbol{E}(\alpha_i^{[2]}\alpha_j^{[2]} \prod_{h\neq i,j} a_h^{[0]})) = a_0^{n-2}a_2^2.$$

The map $\boldsymbol{E}$ in **not** a homomorphism but if $f(\alpha_1,\ldots,\alpha_h),\ g(\alpha_{h+1},\ldots,\alpha_n)$ are in *disjoint variables* we have

$$\boldsymbol{E}(f(\alpha_1,\ldots,\alpha_h)\ g(\alpha_{h+1},\ldots,\alpha_n))$$
$$= \boldsymbol{E}(f(\alpha_1,\ldots,\alpha_h))\boldsymbol{E}(g(\alpha_{h+1},\ldots,\alpha_n))).$$

The basic formula of umbral calculus

$$\boldsymbol{E} \circ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \circ \boldsymbol{E} = \boldsymbol{D} \circ \boldsymbol{E}.$$

The potenziante

Stroh defines the *potenziante* $\pi_{n,g} = \pi_{n,g}(\lambda; a)$ by

$$\pi_{n,g} := \boldsymbol{E}\left((\sum_{j=1}^{n} \lambda_j \alpha_j)^{[g]}\right) = \sum_{\substack{r_1,\ldots,r_n \in \mathbb{N} \\ r_1+\cdots+r_n=g}} \lambda_1^{r_1} \cdots \lambda_n^{r_n} a_{r_1} \cdots a_{r_n}$$

where the $\alpha_1 \ldots, \alpha_n$ are all umbrae and λ some *dual* variables.

We now use the symmetry of the umbrae

The potenziante

In $\pi_{n,g}(\lambda; a)$ the total monomial sum $m_{h_1,\ldots,h_n}(\lambda)$ has as coefficient the product $a_{h_1} a_{h_2} \cdots a_{h_n}$:

$$\pi_{n,g}(\lambda; a) = \boldsymbol{E}\left((\sum_{r=1}^{n} \lambda_r \alpha_r)^{[g]} \right) = \sum_{\substack{h_1 \geq \ldots \geq h_n \geq 0 \\ h_1 + \cdots + h_n = g}} m_{h_1,\ldots,h_n}(\lambda) a_{h_1} a_{h_2} \cdots a_{h_n}.$$

example

With $n = 2$ and $g = 4$ we find

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2)^{[4]}$$

$$= \lambda_1^4\alpha_1^{[4]} + \lambda_2^4\alpha_2^{[4]} + (\lambda_1^3\lambda_2\alpha_1^{[3]}\alpha_2 + \lambda_2^3\lambda_1\alpha_2^{[3]}\alpha_1) + \lambda_1^2\lambda_2^2\alpha_1^{[2]}\alpha_2^{[2]}.$$

Applying $\boldsymbol{E}$ this gives

$$(\lambda_1^4 + \lambda_2^4)a_0a_4 + (\lambda_1^3\lambda_2 + \lambda_2^3\lambda_1)a_1a_3 + \lambda_1^2\lambda_2^2a_2^2,$$

$$= m_{4,0}(\lambda)a_0a_4 + m_{3,1}(\lambda)a_1a_3 + m_{2,2}(\lambda)a_2^2.$$

With $n = g = 3$ we find

$$\boldsymbol{E}((\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3)^{[3]}) =$$

$$m_{3,0,0}(\lambda)a_0^2a_3 + m_{2,1,0}(\lambda)a_0a_1a_2 + m_{1,1,1}(\lambda)a_1^3.$$

Duality

$\pi_{n,g}(\lambda; a) \in \Sigma_{n,g} \otimes \mathbb{C}[a]_{n,g}$ is a *dualizing tensor*

that is, it gives a duality between symmetric functions in n variables of degree g and polynomials in the variables a_i of degree n and weight g.

An elementary but not well known fact

Given two finite dimensional vector spaces U, W and denoting by $U^\vee, W^\vee$ their duals one has the canonical isomorphisms

$$U \otimes W \simeq hom(U^\vee, W) \simeq hom(W^\vee, U).$$

A *dualizing tensor* $\pi \in U \otimes W$ is an element which corresponds, under these isomorphisms, to an isomorphism $U^\vee \simeq W, W^\vee \simeq U$.

Thus, a dualizing tensor π equals, for any basis $u_1, \dots, u_k$ of U, to $\pi = \sum_{i=1}^k u_i \otimes w_i$, where $w_1, \dots, w_k$ is a basis of W.

A basic Formula

From the main Formula of umbral calculus one has

$$\boxed{\sum_{i=1}^{\infty} a_{i-1}\frac{\partial}{\partial a_i}\pi_{n,g} = \boldsymbol{D}\pi_{n,g} = \Big(\sum_{i=1}^{n}\lambda_i\Big)\,\pi_{g-1,n} = \mathrm{e}_1\,\pi_{g-1,n}}$$

The meaning of this formula is that,

using the duality between symmetric functions in n variables and polynomials in the a_i of degree n:

the transpose of the operator $\boldsymbol{D}$ is the multiplication by $\mathrm{e}_1 = \sum_{i=1}^{n}\lambda_i$.

Use the basis by elementary symmetric functions

The potenziante $\pi_{n,g}$ in this basis appears as

$$\pi_{n,g} = \sum_{\substack{0 \le k_1,\dots,k_n \\ k_1+2k_2+\cdots+nk_n=g}} e_1^{k_1} \dots e_n^{k_n} \tilde{U}_{k_1,\dots,k_n},$$

Example

$$\pi_{2,4} = e_1^4 a_0 a_4 + e_2^2 (2a_0 a_4 - 2a_1 a_3 + a_2^2) + e_1^2 e_2 (a_1 a_3 - 4a_0 a_4).$$

A basis of the U-invariants

We get from $\boldsymbol{D}\pi_{n,g} = \left(\sum_{i=1}^{n} \lambda_i\right) \pi_{n,g-1} = \mathrm{e}_1\pi_{n,g-1}$:

$$\boldsymbol{D}\pi_{n,g} = \sum_{\substack{k_1,\dots,k_n \geq 0 \\ \sum i \cdot k_i = g}} \mathrm{e}_1^{k_1} \dots \mathrm{e}_n^{k_n} \boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \sum_{\substack{j_1,\dots,j_n \geq 0 \\ \sum i \cdot j_i = g-1}} \mathrm{e}_1^{j_1+1} \dots \mathrm{e}_n^{j_n} \tilde{U}_{j_1,\dots,j_n}$$

Example

$$\pi_{2,4} = \mathrm{e}_1^4 a_0 a_4 + \mathrm{e}_2^2(2a_0 a_4 - 2a_1 a_3 + a_2^2) + \mathrm{e}_1^2\mathrm{e}_2(a_1 a_3 - 4a_0 a_4).$$

$$\boldsymbol{D}\pi_{2,4} = \mathrm{e}_1^4 a_0 a_3 + \mathrm{e}_1^2\mathrm{e}_2(a_1 a_2 - 3a_0 a_3) = \mathrm{e}_1\pi_{2,3}.$$

A duality for the U-invariants, set $\sum_{i=1}^n \lambda_i = 0$

Definition

Denote by $\overline{\Sigma}_{n,g} \subset \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n] = \mathbb{C}[\lambda_1, \ldots \lambda_n]/(\sum_{i=1}^n \lambda_i)$ the subspace of *symmetric polynomials* in $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$ which are homogeneous of degree g (observe that $\bar{\mathrm{e}}_1 = \sum_{i=1}^n \bar{\lambda}_i = 0$).

This proves the first Theorem

The elements $U_{k_2,\ldots,k_n} := \tilde{U}_{0,k_2,\ldots,k_n}$ form a basis of the space $S_{n,g}$ of the U-invariants of degree n and weight g dual, via $\bar{\pi}_{n,g}$, to the basis $\mathrm{e}_2^{k_2} \ldots \mathrm{e}_n^{k_n}$ of $\overline{\Sigma}_{n,g}$.

The ideas of Stroh

A sketch of the proof

A sketch of the proof

the main idea is to describe, in the duality between symmetric functions and U invariants, the:

space of symmetric functions orthogonal to the space of decomposable elements.

The decomposable elements of $S_{n,g}$ are

$$\sum_{1 \leq h \leq n/2} S_{n,g,h}, \quad S_{n,g,h} := \sum_{j=0}^{g} S_{h,j} \cdot S_{n-h,g-j}.$$

A sketch of the proof

For a given $h \in \mathbb{N}$, $1 \leq h \leq n/2$ we have:

$$(\lambda_1\alpha_1+\cdots+\lambda_n\alpha_n)^{[g]} = \sum_{j=0}^{g}(\lambda_1\alpha_1+\cdots+\lambda_h\alpha_h)^{[j]}(\lambda_{h+1}\alpha_{h+1}+\cdots+\lambda_n\alpha_n)^{[g-j]}$$

which implies the following decomposition of the potenziante:

$$\pi_{n,g}(\lambda_1,\ldots,\lambda_n;a) = \sum_{j=0}^{g}\pi_{h,j}(\lambda_1,\ldots,\lambda_h;a)\cdot\pi_{n-h,g-j}(\lambda_{h+1},\ldots,\lambda_n;a).$$

A sketch of the proof

Consider the ideal $J_h \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ generated by the two elements $\lambda_1 + \cdots + \lambda_h$ and $\lambda_{h+1} + \cdots + \lambda_n$.

Main remark

Modulo this ideal the potenziante $\pi_{n,g}$ becomes a dualizing element between the image of $\overline{\Sigma}_{n,g}$ and $S_{n,g,h}$.

A sketch of the proof

This implies that

the orthogonal to $S_{n,g,h}$ is the space of elements of $\overline{\Sigma}_{n,g}$ which are multiples of the symmetric function

$$p_h := \prod_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} (\bar{\lambda}_{j_1} + \bar{\lambda}_{j_2} + \cdots + \bar{\lambda}_{j_h})$$

A sketch of the proof

1. The orthogonal $O_{n,g}$ to the decomposable elements $\sum_{1\leq h\leq n/2} S_{n,g,h}$ of $S_{n,g}$ is thus the intersection of these orthogonals,
2. so $O_{n,g}$ is the space of elements of $\overline{\Sigma}_{n,g}$ which are multiples of all the symmetric functions p_h, $1 \leq h \leq n/2$
3. but these are irreducible elements in the algebra of symmetric functions so a common multiple is a multiple of their product!

A sketch of the proof

Summarizing

The orthogonal to the decomposable elements $\sum_{1 \leq h \leq n/2} S_{n,g,h}$ of $S_{n,g}$ is the space $O_{n,g}$ of elements of $\overline{\Sigma}_{n,g}$ which are multiples of the product

$$q_n := \prod_h p_h, \quad \deg q_n = 2^{n-1} - 1.$$

This space equals

$$O_{n,g} = q_n \cdot \overline{\Sigma}_{n,g-2^{n-1}+1}$$

$$\dim O_{n,g} = \dim \overline{\Sigma}_{n,g-2^{n-1}+1} = \dim(P_{n,g})$$

from which Stroh's Theorem holds.

This is the main step, the next is to analyze a basis of the complement of this space of multiples.

The final Theorem

This is done by using leading exponents, of polynomials in $\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}$ and proving, by duality, the stated theorem:

For $n \geq 4$ we have the leading exponents *le* of q_n and of $e^{\boldsymbol{n}}$ are the same

$$le = (2^{n-2}, 2^{n-3}, \dots, 2, 1) \quad \text{where } \boldsymbol{n} := (0, 2^{n-4}, 2^{n-5}, \dots, 2, 1, 1).$$

$$q_n = \bar{\lambda}_1^{2^{n-2}} \bar{\lambda}_2^{2^{n-3}} \dots \bar{\lambda}_{n-1} + \text{lower terms}$$

$$e^{\boldsymbol{n}} = \mathrm{e}_3^{2^{n-4}} \mathrm{e}_4^{2^{n-5}} \dots \mathrm{e}_{n-1} \mathrm{e}_n = \bar{\lambda}_1^{2^{n-2}} \bar{\lambda}_2^{2^{n-3}} \dots \bar{\lambda}_{n-1} + \text{lower terms}$$

The final Theorem

Theorem

The elements $U_{\boldsymbol{k}} = U_{k_2,\dots,k_n} = \tilde{U}_{0,k_2,\dots,k_n}$ *with*

$$\boldsymbol{k} \succeq \boldsymbol{n} = (0, 2^{n-4}, 2^{n-5}, \dots, 4, 2, 1, 1)$$

(resp. $\boldsymbol{n} = (0,1)$*) form a basis of a space of perpetuants of degree* $n > 3$ *(resp.* $n = 3$*) and weight* g*.*

The proof is by showing that the products of elementary functions in the dual basis to the $U_{\boldsymbol{k}} = U_{k_2,\dots,k_n}$ for $\boldsymbol{k} \not\succeq \boldsymbol{n} = (0, 2^{n-4}, \dots, 2, 1, 1)$ form a basis of a complement of $O_{n,g}$. This is done by looking at the leading exponents.

Reference

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Perpetuants: A Lost Treasure

Authors: Hanspeter Kraft, Claudio Procesi