Markov numbers, Christoffel words, and the uniqueness conjecture

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Outline

1. Characteristic matrices
   - Basics
   - General properties of $\mu(w)$

2. Frobenius' uniqueness conjecture
   - The map $S : w \mapsto \mu(w)_{1,2}$
   - Tight bounds and uniqueness
   - The Fibonacci and Pell cases

Trace Equals 3 Times Upper Right

Definition

A matrix $M \in SL_2(\mathbb{Z})$ is characteristic if

$$\operatorname{tr} M = 3M_{1,2}.$$

Example

$M = \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$ is characteristic, as

- $\det M = 17 \cdot 13 - 22 \cdot 10 = 1$ and
- $17 + 13 = 3 \cdot 10.$

Simple Constraints

Proposition

Let

$$M = \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix}$$

be characteristic. Elements on the same row or column are coprime, and

$$\alpha^2 \equiv \gamma^2 \equiv -1 \pmod{m}.$$

Also, up to switching α and γ, M is determined by any two elements.

Markov Triples = Characteristic Products

Products of char. matrices need not be characteristic, but:

Theorem

Let M', M'' *be characteristic and* $M = M'M''$. *Then* M *is characteristic* $\iff$ *the upper right elements* m', m'', m *(of* M', M'', M *respectively) verify the Markov equation, i.e.,*

$$(m')^2 + (m'')^2 + m^2 = 3m'm''m.$$

The Morphism $\mu : \{a, b\}^* \longrightarrow SL_2(\mathbb{Z})$

Setting

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

defines an injective morphism $\mu : \{a, b\}^* \longrightarrow SL_2(\mathbb{Z})$.

Note that $\mu(a)$ and $\mu(b)$ are characteristic...

Reversal and μ

Let $\widetilde{w}$ denote the reversal of w.

For instance, if $w = aabab$, then $\widetilde{w} = babaa$.

Lemma

For all $w \in \{a, b\}^*$, $\mu(\widetilde{w}) = \mu(w)^T$.

So, w is a palindrome $\iff$ $\mu(w)$ is symmetric.

Let PAL denote the set of palindromes over $\{a, b\}$.

More Relations on elements

Proposition

Let $w \in \{a,b\}^+$*, and let* $\mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$*. Then* $p > q, r \geq s$*.*

Moreover, $q < r \iff w = \widetilde{u}avbu$ *for suitable* $u, v \in \{a,b\}^*$*.*

Proposition

Let $u \in \mathrm{PAL}$ *and* $\mu(u) = \begin{pmatrix} p & q \\ q & s \end{pmatrix}$*. Then*

$$q + s \leq p \leq 2q + s$$

with $p = q + s \Leftrightarrow u \in a^*$ *and* $p = 2q + s \Leftrightarrow u \in b^*$*.*

Characterizing Characteristic $\mu(w)$

Matrices $\mu(w)$ need not be characteristic; for instance,
$\mu(aa) = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ is not.

Theorem

Let $w \in \{a, b\}^*$. Then

$$\mu(w) \text{ is characteristic} \iff w \in \{a, b\} \cup a\mathrm{PAL}b.$$

A Meaningful Decomposition

Let $\nu : \{a, b\}^* \longrightarrow SL_2(\mathbb{Z})$ be defined by

$$\nu(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \nu(b) = \nu(a)^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is injective and a well-known tool in the study of Christoffel pairs.

It is easy to see that

$$\mu = \nu \circ \zeta$$

where ζ is the injective endomorphism defined by

$$\zeta(a) = ba, \quad \zeta(b) = bbaa.$$

A Consequence

Recall the palindromization map ψ defined by $\psi(\varepsilon) = \varepsilon$ and

$$\psi(vx) = (\psi(v)x)^{(+)} \text{ for } v \in \{a,b\}^*, x \in \{a,b\}$$

where $w^{(+)}$ is the right palindromic closure of w.

Proposition

Let $w = aub$ with $u \in \mathrm{PAL}$. Then

$$\mu(w)_{1,2} = |a\psi(a\zeta(u)b)b|$$

i.e., it is the length of a Christoffel word whose directive word $a\zeta(u)b$ has an antipalindromic middle $\zeta(u)$.

A Similar, Nicer Point of View

The following independent result uses almost the same decomposition:

Theorem (Reutenauer & Vuillon 2017)

For all $v \in \{a, b\}^*$,

$$\mu(a\psi(v)b)_{1,2} = |a\psi(\psi_E(av))b|,$$

where $\psi_E = \theta \circ \psi$ *is the* *antipalindromization* *map,*
and θ *is the Thue-Morse morphism* ($\theta(a) = ab, \theta(b) = ba$).

Definition of S

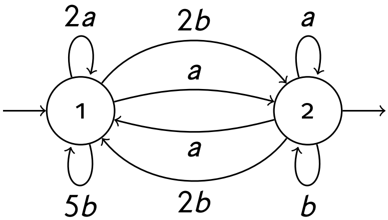
Define a map $S : \{a, b\}^* \longrightarrow \mathbb{N}$ by $S(w) = \mu(w)_{1,2}$.

Since $S(w) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu(w) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, when viewed as a formal series,

$$S = \sum_{w \in \{a,b\}^*} S(w) w$$

is rational.

S as a Series



$$S = (2a + 5b + (a + 2b)(a + b)^*(a + 2b))^* \, (a + 2b)(a + b)^*$$

Not Injective in General

Proposition

For all $u \in \{a, b\}^*$,

- $S(aub) = S(a\widetilde{u}b)$,
- $S(a\theta(u)b) = S(a\theta(\widetilde{u})b)$,

where θ is the Thue-Morse morphism.

Example

- $S(aabb) = 75 = S(abab)$;
- $S(aabbab) = 1130 = S(abaabb)$.

  Thus, even the restriction of S to $\{a, b\} \cup a\mathrm{PAL}b$ is not injective...

What about Christoffel?

Let CH be the set of (lower) Christoffel words over $\{a, b\}$.

Theorem (Borel, Laubie 1993, etc.)

The following holds:

$$\mathrm{CH} = \{a, b\} \cup \left(a\mathrm{PAL}b \cap (\{a, b\} \cup a\mathrm{PAL}b)^2 \right).$$

That is, Christoffel words can be recursively defined:
$w \in \mathrm{CH} \iff w$ *is either a letter or a word of* $a\mathrm{PAL}b$
that is the concatenation of two shorter Christoffel words.

Hence, Markov Triples

We have seen that:

1. $w \in \mathrm{CH} \iff w \in \{a, b\}$ or
   $w \in a\mathrm{PAL}b$ and $w = w'w''$ with $w', w'' \in \{a, b\} \cup a\mathrm{PAL}b$;
2. $w \in \{a, b\} \cup a\mathrm{PAL}b \iff \mu(w)$ is characteristic;
3. Characteristic matrices $M', M'', M'M''$ correspond
   (by their upper right elements) to Markov triples.

Corollary (see Cohn 1972, Reutenauer 2009, etc.)

S maps Christoffel pairs to (nonsingular) Markov triples, i.e., if $w = w'w''$, $w, w', w'' \in \mathrm{CH}$, then

$$S(w')^2 + S(w'')^2 + S(w)^2 = 3S(w')S(w'')S(w).$$

The Conjecture: S Injective on Christoffel

The previous map is actually a bijection, so that the

Conjecture (Frobenius 1913)

Markov triples are uniquely determined by their maximal element.

is equivalent to

Conjecture

The restriction $S|_{\mathrm{CH}}$ is injective.

Evidence Suggesting More Conjectures

Our *limited* experiments *(Markov numbers grow fast!...)* also suggest that:

1. if $w \neq w'$ and $S(w) = S(w')$, then $w, w' \in a\{a, b\}^* b$;
2. if $w \in a\mathrm{PAL}b$ and $S(w)$ is a Markov number, then $w \in \mathrm{CH}$.

Proving Uniqueness via Matrices

A Markov number m is unique if it is the maximal element of a unique Markov triple, or equivalently, if the set

$$S|_{\mathrm{CH}}^{-1}(m) = \{w \in \mathrm{CH} \mid S(w) = m\}$$

is a singleton.

Since μ is injective, and matrices $\mu(w)$ are uniquely determined by any two elements,

$$m \text{ is unique} \iff \exists!\, \gamma : \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix} \in \mu(\mathrm{CH}) \text{ for suitable } \alpha, \beta.$$

(Extended) Known Bounds

Hence, as $\gamma^2 \equiv -1 \pmod{m}$, m is closer to uniqueness when this has few solutions for γ.

Theorem

If $w \in a\mathrm{PAL}b$ and $\mu(w) = \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix}$, then

$$\left(2-\sqrt{2}\right) m < \gamma < \frac{\sqrt{5}-1}{2} m.$$

$$\left\lceil \left(2-\sqrt{2}\right) m \right\rceil \leq \gamma \leq \left\lfloor \frac{\sqrt{5}-1}{2} m \right\rfloor.$$

Tight Versions

The previous bounds are tight, since for example

- $w \in ab^* \implies \gamma = \left\lceil \left(2 - \sqrt{2}\right) m \right\rceil$;
- $w \in a^* b \implies \gamma = \left\lfloor \frac{\sqrt{5}-1}{2} m \right\rfloor$.

Further examples exist, though; for $w = aabab$, e.g., we have
$\gamma = 119 = \left\lfloor \frac{\sqrt{5}-1}{2} 194 \right\rfloor = \left\lfloor \frac{\sqrt{5}-1}{2} m \right\rfloor$.

More Precise Bounds

Theorem

Let $w \in a\mathrm{PAL}b$, $m = \mu(w)_{1,2}$, $\gamma = \mu(w)_{2,2}$. *Then*

$$2m - \sqrt{2m^2 - 1} \leq \gamma \leq \frac{-m + \sqrt{5m^2 - 4}}{2},$$

and the lower (resp. upper) bound is attained if and only if $w \in ab^*$ *(resp.* $w \in a^*b$*).*

Extended Limited Uniqueness Results

Theorem

Let $w = aub, u \in \mathrm{PAL}$ *be such that* $S(w) = 2^h p^k\ [\mu(u)] = 2^h p^k$ *for an odd prime* p *and integers* $h \geq 0$, $k \geq 1$. *(Here* $[M] = M_{1,1} + M_{1,2} + M_{2,1} + M_{2,2}$*.)*
Then for all $w' \in a\mathrm{PAL}b$,

$$w \neq w' \implies S(w) \neq S(w').$$

Remark

It is not known whether there are infinitely many such Markov numbers!

Odd-Indexed Fibonacci & Pell Numbers

Pell numbers are defined by:

- $P_0 = 0, P_1 = 1;$
- $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$.

Well-known: for all $n \geq 0$, $\{1, F_{2n+1}, F_{2n+3}\}$ and $\{2, P_{2n+1}, P_{2n+3}\}$ are Markov triples.

Lemma (cf. Gessel 1972)

A natural number n is an odd- (resp. even-) indexed Fibonacci number if and only if $5n^2 - 4$ (resp. $5n^2 + 4$) is a perfect square.
Similarly, n is an odd- (resp. even-) indexed Pell number if and only if $2n^2 - 1$ (resp. $2n^2 + 1$) is a perfect square.

Corresponding Words and Matrices

In particular,

- $\mu(a^n b) = \begin{pmatrix} 2F_{2n+3} + F_{2n+1} & F_{2n+3} \\ 2F_{2n+2} + F_{2n} & F_{2n+2} \end{pmatrix}$,
- $\mu(ab^n) = \begin{pmatrix} P_{2n+2} & P_{2n+1} \\ P_{2n+1} + P_{2n} & P_{2n} + P_{2n-1} \end{pmatrix}$.

Theorem (Bugeaud, Reutenauer, Siksek 2009)

Odd-indexed Fibonacci and Pell numbers > 5 have no intersection. Also, when written in order, they alternate forming a Sturmian sequence.

Specialized Uniqueness

However, it is not even known whether allodd-indexed Fibonacci and Pell numbers are unique Markov numbers in general!

Theorem

Let $w = a^n b$, $w' \in \mathrm{CH} \setminus \{w\}$, *and* $\gamma' = \mu(w')_{2,2}$.
If $\gamma' > F_{2n+2}$ *or* $\gamma' \leq \frac{2(2-\sqrt{2})}{\sqrt{5}-1} F_{2n+2} \approx 0.948\, F_{2n+2}$, *then*

$$S(w') \neq S(w).$$

Corollary

Let $w = a^n b$, $w' \in \mathrm{CH} \setminus \{w\}$, *and* $\gamma' = \mu(w')_{2,2}$.
If $S(w') = S(w) = F_{2n+3}$, *then* γ' *is not a Fibonacci number.*

Sounds Easy?

The Fibonacci case of the uniqueness conjecture can be restated as follows:

Conjecture

Let $x, y, z \in \mathbb{N}$ *be such that* $x \leq y \leq z$, *with*

$$x^2 + y^2 + z^2 = 3xyz \quad \text{and} \quad \sqrt{5z^2 - 4} \in \mathbb{N}.$$

Then $x = 1$.

Main References

M. Aigner. *Markov's theorem and 100 years of the uniqueness conjecture*. Springer, 2013.

C. Reutenauer. *From Christoffel words to Markoff numbers*. Oxford University Press, USA, 2019.

Thank You

In loving memory of Aldo (1941–2018)

mentor and friend