Wonderful Models for toric arrangements and their cohomology

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Università "La Sapienza"

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All the material in this talk is joint work with Giovanni Gaiffi. It is contained in two papers

De Concini, Corrado; Gaiffi, Giovanni Projective wonderful models for toric arrangements. Adv. Math. 327 (2018), 390-409.

De Concini, Corrado; Gaiffi, Giovanni Cohomology rings of compactifications of toric arrangements. Algebraic & Geometric Topology, (2019), no. 1, 503-532.

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We let k be an algebraically closed field and later $k = \mathbb{C}$, the complex numbers.
We take an algebraic torus $T \simeq (k^*)^n$ over k. We denote by $X^*(T) \simeq \mathbb{Z}^n$ the character group of T.
Take a subgroup $\Gamma \subset X^*(T)$ which is a split direct summand, and a homomorphism $\phi : \Gamma \to k^*$. We define the layer

$$L_{\Gamma,\phi} = \{t \in T |\, \chi(t) = \phi(\chi), \forall\, \chi \in \Gamma\}.$$

Thus a layer is a translate of the subtorus of T intersection of the kernels of the characters in Γ.
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Definition

Suppose we have a finite collection

$$\mathcal{S} = \{(\Gamma_i, \phi_i) | \Gamma_i \in \mathcal{D}, \phi_i \in Hom(\Gamma_i, k^*)\}.$$

The collection of layers

$$\{L_{\Gamma_i, \phi_i}\}_{(\Gamma_i, \phi_i) \in \mathcal{S}}$$

is a toric arrangement if it is closed under the operation of taking connected components of non empty intersections.

Example

In $T = (k^*)^2$, the collection consisting of $L_1 = \{(1, t) | t \in k^*\}$, $L_2 = \{(t^{-2}, t) | t \in k^*\}$, and their two points of intersection $(1, 1)$ and $(1, -1)$ is a toric arrangement.

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Notice that in the definition we have some combinatorial data, the collection of subgroups $\Xi = \{\Gamma_i\}$, and some continuous data, the collections of the elements $\phi_i \in Hom(\Gamma_i, k^*)\}$.
A further piece of combinatorial information about the arrangement is given by the set $\mathcal{M}$ of layers considered as a poset with respect to (reverse) inclusion.
We shall refer to the pair $(\Xi, \mathcal{M})$ as the combinatorial data of the arrangement.

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It appears that toric arrangements have been considered for the first time by Loijenga and then, later, by Procesi and myself over $\mathbb{C}$ in the case in which every layer is a connected component of the intersection of $n-1$ dimensional layers. The main object of study has been the complement

$$\mathcal{A} = T \setminus \cup_{L,layer} L$$

in analogy with what happens with subspace arrangements in projective space. In particular the cohomology groups with complex coefficients of $\mathcal{A}$ have been computed (with different methods both by Loijenga and Procesi and myself).

Recently a number of things related to subspace arrangements has been extended to this case by various people such as Callegaro, D'Adderio, Denham, Delucchi, Migliorini, Pagaria, Suciu and others.

The problem I want to discuss is:

How much of the topology of $\mathcal{A}$ can be detected by the combinatorial data $(\Xi, \mathcal{M})$?

The question is of some interest and has been studied in the analogue case of subspace arrangements.
Here the combinatorial datum is only the poset $\mathcal{M}$ of layers.
I want to recall two results

1. The fundamental group is not combinatorial. Rybnikov gave an example of two combinatorially equivalent arrangements of lines in the projective plane whose complements have different fundamental groups.
2. The homology groups are combinatorial (Goresky-McPherson).
3. The rational homotopy type is combinatorial (as far as I know this was proved by Procesi and myself).

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On the side of toric arrangements, less is known. However if the arrangement is divisorial, that is every layer is a connected component of layers of codimension 1, then one has (Pagaria 2019),

1. The rational cohomology ring $H^*(\mathcal{A}, \mathbb{Q})$ only depends on the poset $\mathcal{M}$.
2. The integral cohomology ring $H^*(\mathcal{A}, \mathbb{Z})$ is not combinatorial.

Before going on, let me explain a little bit the procedure employed to show that rational homotopy type is combinatorial in the case of subspaces.

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Before going on, let me explain a little bit the procedure employed to show that rational homotopy type is combinatorial in the case of subspaces.

The proof is based on the construction of an explicit projective variety X containing $\mathcal{A}$ (the complement of a subspace arrangement) as a dense open set with the property that $X \setminus \mathcal{A}$ is a divisor with normal crossings and smooth irreducible components together with the application of an old result of J. Morgan which explains how to construct out of this a differential graded algebra which is a model for the rational homotopy type of $\mathcal{A}$.

So if we want to follow this path in the toric situation, we have to explicitly add a boundary to $\mathcal{A}$ so to get a smooth projective variety X containing $\mathcal{A}$ as a dense open set with the property that $X \setminus \mathcal{A}$ is a divisor with normal crossings and smooth irreducible components.

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The construction used for subspace arrangement has been extended and axiomatized by Mac Pherson Procesi and then by Li Li for what one calls arrangements of subvarieties.
It is then not hard to see that a toric arrangement is an arrangement of subvarieties in the torus T. It follows (Moci) that one can then construct a smooth variety $\tilde{X}$ with the above properties.
Unfortunately while in the case of subspaces one starts with projective space, here one starts from a torus. It follows that $\tilde{X}$ is not complete so that this is not a solution to our original problem.

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From what I stated, a possible approach to our problem is, given a toric arrangement $\mathfrak{A}$ in a torus T, to first construct a suitable "compactification" Y of T such that

1. Y is smooth.
2. The set of subvarieties whose elements are all the connected components of the intersections of
    1. Closures of the layers in $\mathfrak{A}$.
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We start with the toric arrangement corresponding to a finite collection

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Set $\mathcal{S}' = \{\Gamma_i\} \subset \mathcal{D}$.

Let us give a definition.

Definition

Given a basis B of $X_*(T) = Hom(X^*(T), \mathbb{Z})$, a sublattice $\Gamma \subset X^*(T)$ has equal sign with respect to B if there is a basis $\{\gamma_1, \ldots, \gamma_r\}$ of Γ such that $\langle \gamma_i, b \rangle \geq 0$ for all $i = 1, \ldots r$, $b \in B$.

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We then fix a basis of $X^*(T)$ thus getting a identification of $X_*(T)$ with $\mathbb{Z}^n$ and of $V := Hom(X^*(T), \mathbb{R}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^n$.

The decomposition of V into quadrants gives a fan $\mathcal{F}_0$ whose corresponding T-variety is $(\mathbb{P}^1)^n$. One immediately sees that his "compactification" of T is no good for our purposes unless each Γ_i is spanned by a subset of B.

However we give a completely explicit recursive procedure which produces a fan $\mathcal{F}$ with the following properties.

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1. $\mathcal{F}$ is obtained from $\mathcal{F}_0$ by successive subdivisions of 2-dimensional faces. This corresponds to successively blowing up T orbit closures of codimension 2.
2. $\mathcal{F}$ is smooth and projective.
3. Every n dimensional cone C in $\mathcal{F}$, is the cone of non negative linear combinations of a basis with respect to which every $\Gamma_i \in \mathcal{S}'$ has equal sign.

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Consider now the T-variety Y associated to the fan $\mathcal{F}$. Fix a layer $L_{\Gamma,\phi}$ in our toric arrangement. We know that there is a subtorus $H \subset T$ such that $L_{\Gamma,\phi}$ is a translate of H. The restriction homomorphism $r : X^*(T) \to X^*(H)$ whose kernel is Γ induces an inclusion, of $V_H = Hom(X^*(H), \mathbb{R})$ into V.

Theorem

1. *V_H is a union of cones of the fan $\mathcal{F}$. In particular we get a smooth fan $\mathcal{F}_H$ in V_H.*
2. *Denote by $\overline{L}_{\Gamma,\phi}$ the closure of the layer $L_{\Gamma,\phi}$ in Y. $\overline{L}_{\Gamma,\phi}$ is a H-variety whose fan is $\mathcal{F}_H$. In particular $\overline{L}_{\Gamma,\phi}$ is smooth.*
3. *Let $\mathcal{O}$ be a T orbit in Y and C the corresponding cone. Then $\overline{O} \cap \overline{L}_{\Gamma,\phi} \neq \emptyset$ iff $C \subset V_H$. If this is the case $\overline{O} \cap \overline{L}_{\Gamma,\phi}$ is the closure of the H orbit in $\overline{L}_{\Gamma,\phi}$ whose corresponding cone is C.*

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As a consequence of this theorem we get

Proposition

The arrangement $\mathfrak{A}'$ of subvarieties consisting of

(i) *The closure in Y of the layers in the toric arrangement $\mathfrak{A}$.*

(ii) *The closures of T orbits in $Y \setminus T$ and their intersections with layer closures.*

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At this point one follows the standard procedure and gets a projective X such that

1. X contains the complement $\mathcal{A} = T \setminus \cup L_{\Gamma_i,\phi_i}$ as a dense open set.
2. $X \setminus \mathcal{A}$ is a divisor with normal crossings and smooth irreducible components.

Thus one may try to compute the Morgan algebra for this variety with the divisor $X \setminus \mathcal{A}$ and get information on the rational homotopy type of $\mathcal{A}$.

An Example.

Let us take our usual arrangement consisting of $L_1 = \{(1,t)|t \in k^*\}$, $L_2 = \{(t^{-2},t)|t \in k^*\}$, and their two points of intersection $(1,1)$ and $(1,-1)$.

In this case the fan of the variety Y given by our construction is

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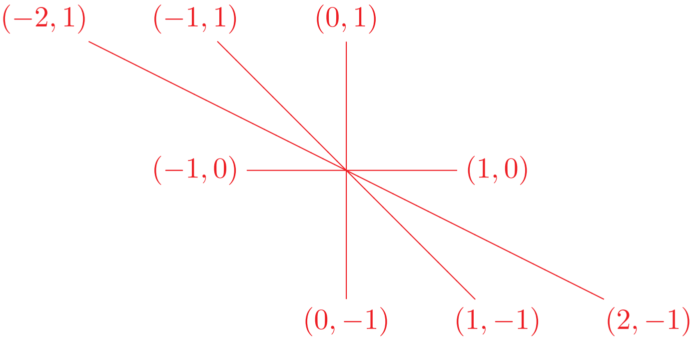

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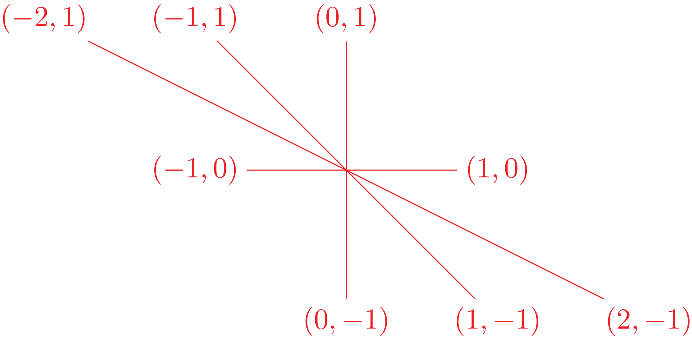

So far so good. In the second paper we performed all the necessary cohomology computation for the cohomology of X and all the partial intersections of the irreducible components of the divisor $X \setminus \mathcal{A}$.

So we can explicitly write down the Morgan model $\mathcal{M}_X$ for the the rational homotopy type of $\mathcal{A}$. In particular the cohomology algebra of $\mathcal{M}_X$ is the cohomology algebra with rational coefficients of $\mathcal{A}$.

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How to get rid of Y

We need to get rid of Y. The ideal is take all Y's at the same time.
To be precise, let us consider the set of all toric varieties Y whose fan $\mathcal{F}_Y$ satisfies the properties

1. $\mathcal{F}$ is obtained from $\mathcal{F}_0$ by successive subdivisions of 2-dimensional faces. This corresponds to successively blowing up T orbit closures of codimension 2.
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Set $Y > Y'$ if there is a (necessarily unique) T equivariant morphism $\pi : Y \to Y'$ extending the identity on T. This is equivalent to ask that each cone of $\mathcal{F}_{Y'}$ is a union of cones in $\mathcal{F}_Y$.

For each Y we construct the corresponding X_Y and if $Y > Y'$, $\pi : Y \to Y'$ induces a map $\overline{\pi} : X_Y \to X_{Y'}$ which in turn induces a differential graded algebra homomorphism

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The description of $\mathcal{M}(\mathcal{A})$

As before let $V := Hom(X^*(T), \mathbb{R}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
The space $L(T)$ is the space of functions

$$f : V \to \mathbb{R}$$

such that

1. f is continuous.
2. there is a complete smooth fan $\mathcal{F}$ such that
   1. the restriction of f to each cone of $\mathcal{F}$ is linear.
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We take the ring $\mathcal{C}(T)$ of continuous functions on V generated by $L(T)$. We now take the algebra

$$\mathcal{C}(T) \otimes \bigwedge(L(T)).$$

A function $f \in L(T)$ will be denoted by f when considered in $\mathcal{C}(T)$, by τ_f when considered in $\bigwedge[L(T)]$
We introduce a grading by setting

$$deg\, f = 2,\, deg\, \tau_f = 1.$$

a differential d of degree 1 by setting

$$d(\tau_f) = f,\, d(f) = 0.$$

We now take the the ideal I generated by

1. $X(T)^* \otimes \mathbb{Q} \subset L(T)$ considered as linear functions on V with rational values on $X_*(T)$.
2. The elements
$$P\tau_{f_1} \cdots , \tau_{f_h}$$
such that $Pf_1 \cdots f_h = 0$.

Notice that I is graded and it is preserved by the differential. We then define

$$\mathbb{B}(T) = (\mathcal{C}(T) \otimes \bigwedge(L(T))/I.$$

we have not yet mentioned our toric arrangement.

We start with toric arrangement corresponding to a finite collection

$$\mathcal{S} = \{(\Gamma_i, \phi_i) | \Gamma_i \in \mathcal{D}, \phi_i \in Hom(\Gamma_i, k^*),\ i = 1, \dots m\}.$$

For every pair (Γ_i, ϕ_i) we take a variable t_i and an exterior variable τ_i and we consider the Weil algebra $W = \mathbb{Q}[t_i] \otimes \bigwedge(\tau_i)$ with obvious structure of a differential graded algebra, that is deg$t_i = 2$, deg$\tau_i = 1$, $d(\tau_i) = t_i$, $d(t_i) = 0$.
We take the differential graded algebra $\mathcal{W} := \mathbb{B}(T) \otimes W$.
In order to proceed we need to quotient a suitable ideal J, whose description is rather complicated.

For $v \in V$, $\lambda \in \Gamma$, we set

$$\pi_\lambda(v) = min(0, \langle v, \lambda \rangle)$$

To simplify notation we denote by G_i the layer corresponding to the pair (Γ_i, ϕ_i). Given G_i we set $B_i = \{k | G_k \subset G_i\}$
We fix a pair (i, A), $A \subset \{1, \dots m\}$ such that,

$$\text{if } j \in A,\ G_j \supsetneq G_i. \tag{1}$$

There a unique unique connected component M (a layer) of $\cap_{j \in A} G_j$ containing G_i. M corresponds to a pair (Γ, ϕ), $\Gamma \subset \Gamma_i$.
We choose vectors $\psi_1, \dots, \psi_s$ in Γ_i inducing a basis of Γ_i / Γ
Now we can split A as a disjoint union $A_1 \cup A_2$ and introduce the elements.

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$$G(i,A_1,A_2)=\prod_{u=1}^{s}(\sum_{l\in B_i}t_l-\pi_{\psi_u})\prod_{j\in A_1}t_j\prod_{k\in A_2}\tau_k,\ \ s\neq 0$$

,

$$G(i,A_1,A_2)=\prod_{j\in A_1}t_j\prod_{k\in A_2}\tau_k,\ \ s=0.$$

The ideal J generated by:

- $F\tau_{f_1}\cdots\tau_{f_s}t_{i_1}\cdots t_{i_h}\tau_{j_1}\cdots\tau_{j_p}$
  with $F\in C(T)$, $f_1,...,f_s\in L(T)$ and

$$\mathrm{supp}(F\ f_1\cdots f_s)\bigcap(\cap_{r=1}^{h}\Gamma_{i_r}^{\perp})\bigcap(\cap_{m=1}^{p}\Gamma_{j_m}^{\perp})=\emptyset.$$

- $G(i,A_1,A_2)$, for every pair $(i,A_1\dot{\cup}A_2)$ such that if $j\in A_1\dot{\cup}A_2$ then $G_i\subsetneq G_j$.

Theorem

1. *The ideal $J \subset \mathcal{W}$ does not depend on the choice of the bases ψ.*
2. *The ideal J is homogeneous and preserved by the differential.*
3. *The algebra $\mathcal{M}(\mathcal{A}) = \mathcal{W}/J$ is a model for the rational homotopy type of $\mathcal{A}$ which in particular depends only on the combinatorial data $(\Xi, \mathcal{M})$.*