On the commutative equivalence of bounded context-free and regular languages

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Regularity conditions for languages

Conditions that guarantee that a language is accepted by a finite automaton

[A. de Luca, S. Varricchio, 1999, EATCS]

"Finiteness and Regularity in Semigroups and Formal Languages"

A. de Luca, S. Varricchio, 1999, EATCS

BURNSIDE PROBLEM FOR SEMIGROUPS

Study of the conditions that guarantee that a finitely generated and periodic semigroup is finite

FINITENESS CONDITIONS VS LANGUAGES

Via the syntactic semigroup of L, a finiteness condition is a regularity condition for L

Brzozowski Conjecture, 1969

For $n, k \geq 1$, $B(k, n, n+1)$ is the free semigroup over k generators in the variety $x^n = x^{n+1}$

$$\varphi : A^* \longrightarrow B(k, n, n+1)$$

$$s \in B(k, n, n+1) \implies \varphi^{-1}(s) \in \mathsf{Rat}(A^*)$$

[A. de Luca, S. Varricchio, 1990]

Positive solution for $n \geq 5$

The Word problem is recursively decidable for $n \geq 5$

The Permutation property for semigroups

[A. Restivo, Ch. Reutenauer, 1984]

S finitely generated and periodic semigroup. Then S is finite if and only if S is permutable

[M. Curzio, P. Longobardi, M. Maj, D. Robinson, 1983, 1985]

Characterization of permutable groups

A proof of Restivo and Reutenauer result

[Ch. 3, Prop. 3.4.1]

Let S be a finitely generated and permutable semigroup

The set of the canonical representatives of S is a bounded language

Bounded languages

Definition

Let $L \subseteq A^*$. L is called n-bounded if there exist n words $u_1, u_2, \ldots, u_n$ such that

$$L \subseteq u_1^* u_2^* \cdots u_n^*$$

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$$L \subseteq \{u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n} \ : \ k_1, \ldots, k_n \in \mathbb{N}\}$$

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L is called bounded if it is n-bounded for some n

A characterization of bounded languages

[Ch. 2, Theorem 2.5.1, A. Restivo, Ch. Reutenauer, 1983]

L is bounded if and only if there exists some $n \geq 1$ such that

the set of factors of L does not contain n-divided words

Bounded languages

in

the theory of context-free languages

Main result

Every bounded context-free language L_1 is commutatively equivalent to a regular language L_2

Commutative Equivalence

Let $L_1, L_2 \subseteq A^*$

L_1 is commutatively equivalent to L_2 if there exists a bijection

$$f : L_1 \longrightarrow L_2$$

such that, for every $u \in L_1$,

$$\psi(u) = \psi(f(u))$$

The Parikh morphism

- $A = \{a_1, \ldots, a_t\}$
- $\psi : A^* \longrightarrow \mathbb{N}^t$
- $\forall u \in A^*, \quad \psi(u) = (|u|_{a_1}, |u|_{a_2}, \ldots, |u|_{a_t})$

Schützenberger Conjecture

(Schützenberger, 1956)

Every finite maximal (unique factorization, variable-length) code is commutatively equivalent to a prefix code

- Bounded and sparse context-free languages

- Our problem and its relations with the theory of formal languages

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Sparse languages

$L \subseteq A^*$

The counting function of L is the map

$$c_L : \mathbb{N} \longrightarrow \mathbb{N}$$

such that

$$c_L(n) = \mathrm{Card}(L \cap A^n)$$

Sparse and bounded languages

Definition

L is sparse or poly-slender if $c_L(n)$ is upper bounded by a polynomial in n

Sparse and bounded languages

Theorem (Latteux and Thierrin 1984; Ibarra and Ravikumar 1986; Raz 1997; Ilie, Rozenberg and Salomaa 2000)

A context-free language is sparse if and only if it is bounded

Sparse and bounded languages

Theorem (D., Intrigila, and Varricchio 2006)

Let L be a bounded context-free language over the alphabet A

Then there exists a regular language L' over an alphabet B such that, for all $n \geq 0$,

$$c_L(n) = c_{L'}(n)$$

The Problem

Given a bounded context-free language L, does it exist a regular language L' which is commutatively equivalent to L?

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Given a bounded context-free language L, does it exist a regular language L' which is commutatively equivalent to L?

Do there exist a regular language L' over the same alphabet of L and a bijection

$$f : L \longrightarrow L'$$

such that, for every $u \in L$, u and $f(u)$ have the same Parikh vector ?

- L commutatively equivalent to L' implies

$$\forall\, n \in \mathbb{N}, \quad c_L(n) = c_{L'}(n)$$

Main result

Theorem (D., Intrigila, 2014)

Every bounded context-free language is commutatively equivalent to a regular language. Moreover such construction is effective

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More generally, we prove that:

Theorem

Every bounded semi-linear language is commutatively equivalent to a regular language. Moreover such construction is effective

Example

$$L \subseteq a^*\mathbf{b}a^*$$

$$L = \{a^{1+x_1+2x_2}\mathbf{b}a^{x_2} \ : \ x_1, x_2 \geq 0\}$$

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$$a^{1+x_1+2x_2}\mathbf{b}a^{x_2} \quad \xrightarrow{f} \quad a^{1+x_1}\mathbf{b}a^{3x_2}$$

A geometrical perspective

$$a^x \mathbf{b} a^y \longrightarrow (x, y) \in \mathbb{N}^2$$

Then the language

$$L = \{a^{1+x_1+2x_2} \mathbf{b} a^{x_2} \ : \ x_1, x_2 \geq 0\}$$

becomes

$$\{(1 + x_1 + 2x_2, \quad x_2) \ : \ x_1, x_2 \geq 0\}$$

A geometrical perspective

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and the language

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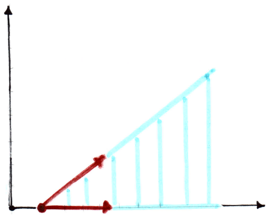


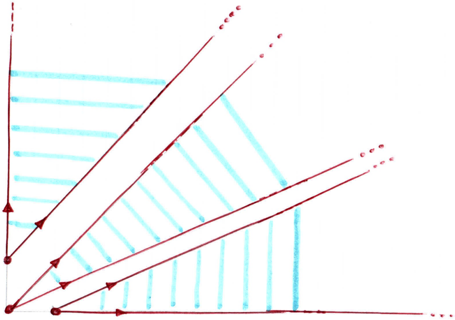
$(1 + x_1 + 2x_2,\ x_2)$

$(1 + x_1,\ 3x_2)$

$$L \subseteq a^*\mathbf{b}a^*$$

$$L = L_1 \cup L_2 \cup L_3$$

1. $$L_1 = a^{1+x_1+2x_2}\mathbf{b}a^{x_2}$$
2. $$L_2 = a^{2y_1+y_2}\mathbf{b}a^{y_1+2y_2}$$
3. $$L_3 = a^{z_1}\mathbf{b}a^{1+z_2+2z_1}$$



1. $L_1 = a^{1+x_1+2x_2}\mathbf{b}a^{x_2} \longrightarrow L'_1 = a^{1+x_1}\mathbf{b}a^{3x_2}$

2. $L_2 = a^{2y_1+y_2}\mathbf{b}a^{y_1+2y_2} \longrightarrow L'_2 = a^{3y_1}\mathbf{b}a^{3y_2}$

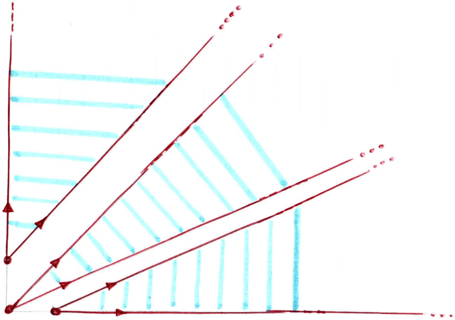
3. $L_3 = a^{z_1}\mathbf{b}a^{1+z_2+2z_1} \longrightarrow L'_3 = a^{3z_1}\mathbf{b}a^{z_2+1}$

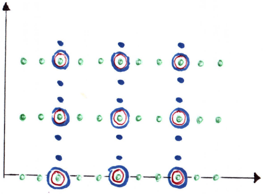
1. $L_1 = a^{1+x_1+2x_2}\mathbf{b}a^{x_2} \longrightarrow L_1' = a^{1+x_1}\mathbf{b}a^{3x_2}$

2. $L_2 = a^{2y_1+y_2}\mathbf{b}a^{y_1+2y_2} \longrightarrow L_2' = a^{3y_1}\mathbf{b}a^{3y_2}$

3. $L_3 = a^{z_1}\mathbf{b}a^{1+z_2+2z_1} \longrightarrow L_3' = a^{3z_1}\mathbf{b}a^{z_2+1}$

Obstruction : $a^3\mathbf{b}a^3 \in L_1' \cap L_2' \cap L_3'$





A regular language equivalent to $L = L_1 \cup L_2 \cup L_3$

$$
\begin{array}{ll}
(3x_1+1, \quad 3x_2) & \cup \\
(3x_1+1, \quad 3x_2+1) & \cup \\
(3x_1+1, \quad 3x_2+2) & \cup \\
(3y_1, \quad 3y_2) & \cup \\
(3z_1, \quad 3z_2+1) & \cup \\
(3z_1, \quad 3z_2+2) & \cup \\
(3z_1+2, \quad 3z_2+1) & .
\end{array}
$$

Some elements of the solution

Ambiguities of context-free languages

1. $L \subseteq u_1^* u_2^* \cdots u_k^*$ context-free bounded

2. Ambiguity of L as a context-free language

3. Ambiguity of L as a subset of the product

$$u_1^* \cdots u_k^*$$

Techniques

1. Faithful linear representation of bounded languages:
   Ginsburg and Spanier, 1966; Eilenberg Cross-section, 1974

2. Elementary number theory (on semi-linear sets):
   Eilenberg and Schützenberger result on semi-simple sets, 1969

3. Geometrical decomposition of semi-linear sets

4. Combinatorics of variable-length codes

Open problems

Gap Theorem

Theorem (Incitti 1999, Bridson and Gillman 1999)

A context-free language is either sparse or of exponential growth

Gap Theorem

Theorem (Incitti 1999, Bridson and Gillman 1999)

A context-free language is either sparse or of exponential growth

Theorem (Grigorchuk, Machí, 1998)

Existence of languages of intermediate growth

$$L = \{a^{n_1} b a^{n_2} b \cdots a^{n_{k-1}} b a^{n_k} \ : \ n_1 \leq n_2 \leq \cdots \leq n_k, \ k \geq 1\}$$

Languages of exponential growth

there exist languages of exponential growth that are not commutatively equivalent to regular languages

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Search of conditions for and characterizations of languages of exponential growth commutatively equivalent to regular ones.

[D., Carpi, 2018, 2019]

Finite-index context-free languages

Languages of exponential growth

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Finite-index context-free languages

Minimal linear grammars

Minimal linear grammars and the commutative equivalence problem

Thank you